Adiabatic destruction of Anderson localization

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We study the diffusive wave spreading in a one-dimensional Anderson model due to a slow parameter variation with a frequency ω. The diffusion rate depends on frequency in a power law $D \propto \omega^{\alpha}$. A theoretical approach based on Mott's mechanism of energy absorption gives $\alpha = \frac{2}{3}$ in agreement with numerical data. The same results are also found for a kicked rotator model of quantum chaos.

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Recently many efforts have been devoted to the study of the behavior of quantum systems under a slow parameter variation. In the domain of quantum chaos the level dynamics with respect to a parameter variation has been investigated [1] and a universal correlation for level velocities has been found [2]. This theory can be applied for dynamical models and for noninteracting electrons in a random potential in the metallic regime, as well. Such investigations are also important for understanding the response to slow time variation of external electromagnetic fields. For electrons in random potentials one of the main related characteristics is the dependence of conductivity on the external field frequency. In the limit of small fields the Kubo formalism can be applied to compute the ac conductivity $\sigma_{\rm ac}(\omega)$. For localized states the dependence of conductivity on small frequencies was derived by Mott (see, e.g., [3]). Physically $\sigma_{\rm ac}$ characterizes the energy absorption under the influence of an external electric field. However, the time variation of the electric field can also lead to the destruction of localization in one-dimensional disordered chains and to the appearance of electron diffusion along the chain itself. These effects are usually treated by using a perturbative approach in the limit where the fields go to zero. Then the question arises as to what the result will be for a finite field amplitude. This problem constitutes the main subject of our paper.

Here, we investigate the destruction of Anderson localization in one dimension under the slow variation in time of the external field or some other suitable parameter. To the best of our knowledge this question has been addressed in different contexts in [4,5]. In [4] the destruction of dynamical localization was investigated for the kicked rotator model with modulated amplitude. The numerical experiments showed that slow parameter modulation with frequency ω leads to diffusion along the chain with a diffusion rate $D \propto \omega^{\alpha}$, where $\alpha \approx 0.6$. However, the theoretical consideration given in [4] was not able to derive the numerical value of α. Another theoretical study of a related problem [one-dimensional (1D) localized electron in an external ac field] was approached in [5] where the exponent $\alpha = 1$ was found, in disagreement with the numerical value of [4]. This apparent contradiction stimulated our research on the phenomenon discussed.

The adiabatic destruction of Anderson localization was investigated in two models. The first one is the 1D Anderson chain with modulated hopping elements

$$i\dot{\psi}_n = E_n\psi_n + V(1+\epsilon(t))(\psi_{n+1}+\psi_{n-1}) \ . \qquad (1)$$

Here, E_n are energies randomly distributed in the interval $[-W,W]$, $\epsilon(t) = \epsilon(\sin(\omega_1 t) + \sin(\omega t))$ characterizes the modulated hopping, and ω,ω_1 are two incommensurate frequencies. For $\epsilon = 0$ all eigenstates are exponentially localized with a localization length given by $l \approx 25(V/W)^2 > 1$ for $E \approx 0$ [8]. We chose two frequencies since in this case for $\omega \sim \omega_1 \sim 1$ the problem can be effectively reduced to a localization problem in two dimensions [6,7,4]. In this case the localization length l_2 generally grows exponentially as $\ln l_2 \sim l$, and the time t^* at which the localization takes place is also exponentially large. For shorter times $t \ll t^*$ the spreading of wave packets over the lattice goes diffusively with a diffusion rate $D = (\Delta n)^2/t$. Our main task will be to determine the dependence of the diffusion rate on the small frequency ω. This dependence also allows one to determine the diffusion rate in the case of slow nonmonochromatic perturbation (noise) with $d\epsilon/\epsilon dt$ playing the role of the typical frequency ω.

Our theoretical understanding of the frequency dependence of the diffusion rate along the chain is based on the following mechanism. The time variation of the external field leads, as in Mott's picture, to a diffusive growth of the electron energy with a diffusion rate $D_E = (\Delta E)^2/t$. This energy absorption leads to a spreading in the electron momentum $\Delta p \sim \Delta E/V \sim \sqrt{D_E t}/V$, since in (1) the dispersion law, for small disorder, is given by $E = 2V\cos p$. Therefore the width of the spreading along the lattice grows as $\Delta n \sim V\Delta p\, t \sim \sqrt{D_E}\, t^{3/2}$. The time t_c during which the spreading remains coherent can be estimated from the condition $\Delta n \sim l$. After this time the coherence is destroyed and random transitions between localized states take place. The size of one jump is of

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the order of l and the transition time is $t_c \sim (l^2/D_E)^{1/3}$. Since these transitions occur in an uncorrelated way they lead to a diffusion along the chain with the rate

$$D \approx l^2/t_c \sim l(lD_E)^{1/3} . \quad (2)$$

To obtain the final estimate for D we should use the real expression for the diffusion rate in energy D_E. This can in turn be estimated on the basis of Mott's picture of energy absorption [3] due to transitions between quasidegenerate double-hump states. The diffusion rate is $D_E \sim \omega^2\Gamma$ with the transition rate $\Gamma \sim F^2\rho$, $\rho \sim l/V$ being the density of localized double-hump states and F the matrix element for one-photon transition. For the model (1) $F \sim \epsilon V$ and so

$$\frac{D}{l} \sim (\epsilon\omega l\sqrt{V})^{2/3} . \quad (3)$$

Of course, in the above estimate the parameter ϵ cannot be too small. Indeed the coherence time t_c should be larger than the transition time $1/\Gamma$ [9], which gives $\epsilon > \sqrt{\omega/V}/l$. Another more restrictive condition is that $\Gamma\rho > 1$ which characterizes the nonperturbative regime. This gives a condition which is independent of frequency: $\epsilon > 1/l$.

It is interesting to note that if the perturbation is characterized by a continuous spectrum (adiabatic noise) with width ω then the diffusion rate D_E is proportional to the noise intensity ν. In such a formulation $D \sim \nu^\beta$, with $\beta = \alpha/2 = 1/3$, which is the same as the effect of noise in quasimomentum analyzed by Cohen [10] in the kicked rotator model. The important feature of such kind of noise is that, for small ν, it gives a diffusion rate much bigger than the rate for the standard type of noise ($D \sim \nu$ [11]). In spite of some analogy with [10] we should stress that the noise considered there was not adiabatic and had a very specific form, so that its physical applications were quite restrictive and not obvious. The condition $\Gamma\rho > 1$ which is necessary for two frequencies in (1) in order to have a large 2D localization length might be unnecessary in the presence of adiabatic noise. Let us mention that our mechanism, which leads to $\alpha = 2/3$, is quite different from the approach [5] which gives $\alpha = 1 > 2/3$. This means that at small frequencies the diffusion rate (3) is dominating.

We tested the theoretical prediction (3) in numerical simulations of the model (1). The typical example of spreading over the lattice is presented in Fig. 1. The width of the wave packet grows diffusively with time, $(\Delta n)^2 = Dt$, which allows one to determine the diffusion rate D. We studied the dependence of D on the different model parameters. The results are presented in Fig. 2. The ratio between the two frequencies was kept fixed, $\omega_1/\omega = 1.618\ldots$. Initially the energy was chosen in the middle of the band ($E = 0$), and we fixed $\epsilon = 0.5$ [12] and $V = 1$. The parameter ranges were $2 \times 10^{-4} \leq \omega \leq 0.6$, $0.7 \leq W \leq 1.9$, so the localization length was changing by one order of magnitude. The diffusion rate obtained varies in the interval 3×10^{-2}–42. After the rescaling of variables according to (3) all data are approximately described by $D \approx 0.36l(\omega l)^\alpha$. The least squares fit gives

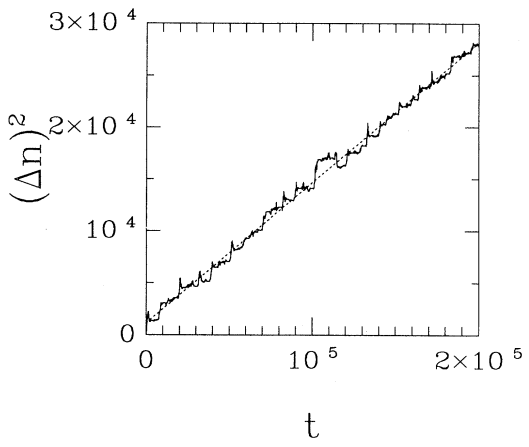

FIG. 1. Dependence of the square width of the wave packet as a function of time for the model (1), with $W = 1.5$, $\omega = 0.001$, $\omega_1/\omega = 1.618\ldots$, $V = 1$, and $\epsilon = 0.5$. The size of the lattice is $N = 2048$. Initially, only one site level is excited with energy $E = 0$. The dashed line shows the linear fit with $D = 0.135$.

$\alpha = 0.74 \pm 0.02$. This value is quite close to the theoretical value 2/3. However, a 10% difference is evident. The value $\alpha/2 = 0.37$ is in agreement with the numerical results $0.35 < \beta < 0.38$ of [10] and we think that this difference is connected with the effect of quantum correlations, as discussed in [10]. For $\omega l > 1$ saturation takes place (see Fig. 2) and the diffusion rate D becomes comparable with the classical diffusion rate $D_0 \sim l$. This happens when the coherence time becomes smaller than the localization time $t_c < l/V$, from which it follows that $\epsilon\omega l > V$.

We also studied the adiabatic destruction of coherence in the kicked rotator model [13]. In this model the evo-

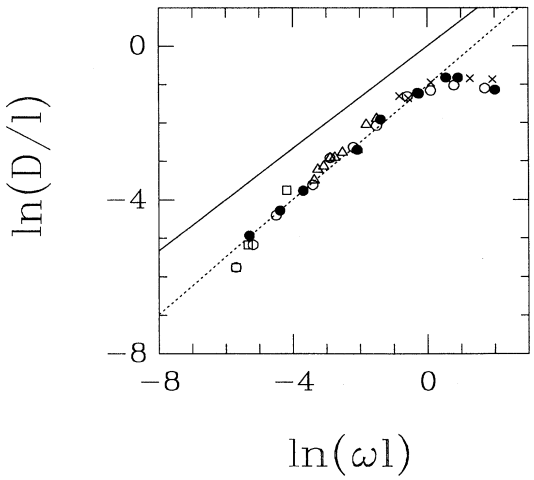

FIG. 2. Diffusion rate D as a function of the rescaled variables for the model (1) with $V = 1$, $\epsilon = 0.5$, $l = 25(V/W)^2$, $\omega_1/\omega = 1.618\ldots$, and $E = 0$. Symbols are ○ for $W = 1.5$, • for $W = 1$, × for $\omega = 0.07$, △ for $\omega = 0.005$, and □ for $\omega = 0.0003$. Dashed line represents the least squares fit while the full line shows the theoretical slope $\alpha = 2/3$.

lution operator is

$$U = \exp(-iTn^2/2)\exp[-ik(t)\cos\theta], \tag{4}$$

where $k(t) = k(1 + \epsilon(\cos(\omega t) + \sin(\omega_1 t)))$ and t is measured in the number of kicks. For $\epsilon = 0$ this is the standard kicked rotator model in which chaotic diffusive excitation takes place for $K = kT > 1$, and the quasiclassical limit corresponds to $k \gg 1, T \ll 1$, with $K =$ const. The diffusion rate is $D_0 \approx k^2/2$ for $K \gg 1$. Quantum interference leads to a localization of this diffusion with a length $l \approx D_0/2$. The model (4) with one frequency has been studied in [4]. Here we have chosen two frequencies since in this case the model is effectively equivalent to a 3D solid state problem [7], and unlike in [4] the localization effects are much less pronounced.

The estimate (3) should be slightly modified to be applied to (4). The dispersion law is still given by $E = 2k\cos p$, where $p = \theta$ plays the role of momentum and the estimate (2) holds. However, now the diffusion rate in quasienergy is $D_E \sim (\epsilon\omega l)^2$ since for double-hump states the matrix element $F \sim \epsilon k$ [14] and the density $\rho \sim l$, since the quasienergies are homogeneously distributed in $[0, 2\pi]$. This gives

$$\frac{D}{D_0} \sim (\epsilon\omega D_0^{3/2})^{2/3} . \tag{5}$$

Our theoretical explanation is different from the arguments presented in [4], where the exponent α was assumed to be connected with the level statistics for states in one localization length interval. To test this connection we slightly modified (4) by adding a second harmonic $(k/2)\sin(2\theta)$ in the kick potential and a magnetic flux ϕ in the free rotation term $[n \to (n + \phi)]$. In this way the system changes universality class, from GOE to GUE [15]. However, our numerical calculations show that this modification does not affect the exponent α which is in agreement with (5).

Our numerical results for the diffusion rate dependence on the parameters of (4) are presented in Fig. 3. The parameters ranges were $3 \times 10^{-5} < \omega < 1$, $3 < k < 16.8$, $2 \times 10^{-2} < \epsilon < 0.5$, $kT = 6$. The diffusion rate obtained was in the range 9×10^{-3}–1.3×10^{2}. The basis was varying up to 8192 levels. The rescaling of variables confirms the scaling relation (5). The usual best fit procedure gives $D \approx 1.8D_0(\epsilon\omega D_0^{3/2})^{\alpha}$ with $\alpha = 0.73 \pm 0.02$. The value of the exponent α is in agreement with the numerical value obtained for the Anderson model. This confirms, once more, that the kicked rotator represents a physically meaningful model and the results obtained from it can be used to explore solid state physics too. Let us note that both the numerical value obtained for α and the scaling relation (5) are slightly different from those obtained in [4]. We think that the choice of two driving frequencies instead of one allows us to investigate a much wider range of parameters, which in turn leads to a better fitting of the data.

The results obtained above allow us to find the diffusion rate along the lattice for the more physical problem of electrons in a random potential under the influence

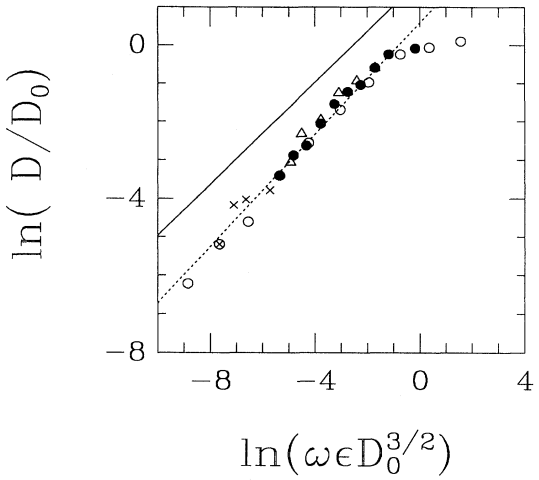

FIG. 3. Diffusion rate D as a function of the rescaled variables for the model (4) with $kT = 6$, $\omega_1/\omega = 1.618...$, and $D_0 = k^2/2$. Symbols are ○ for $k = 3$, $\epsilon = 0.5$; • for $\omega = 0.001$, $\epsilon = 0.5$; × for $\omega = 0.0001$, $\epsilon = 0.5$; △ for $\omega = 0.001$, $k = 10.08$. Dashed line represents the least squares fit while the full line shows the theoretical slope $\alpha = 2/3$.

of a low frequency electric field $\mathcal{E}(t)$. For electrons with the dispersion law $E = V\cos p$ the mechanism discussed above gives the estimate (2). The diffusion rate in energy is still given by the Mott formula but now the matrix element is $F = \langle m|\mathcal{E}(t)\hat{x}|m'\rangle \sim \mathcal{E}l\ln(V/\omega)$ for the 1D case. Here $\mathcal{E}$ is the electric field amplitude and we use units in which $\hbar = e = a = 1$, where a is the lattice spacing. Then it follows that

$$D \approx \left(\frac{\omega^2\mathcal{E}^2 l^7 \ln^2(V/\omega)}{V}\right)^{1/3} . \tag{6}$$

We note that the diffusion rate along the lattice obtained in the limit of small frequencies decreases much more slowly than the usual Mott rate for energy absorption $D_E \propto \omega^2$.

In deriving (6) we implicitly assumed that the driving field is not monochromatic but contains at least two frequencies of order ω. Another possibility is to use a continuous field spectrum with a frequency width ω. The condition for the applicability of (6) for two driving frequencies is $\Gamma\rho > 1$, which gives $\mathcal{E} > V/[l^2\ln(V/\omega)]$. For adiabatic noise the condition is $t_c > 1/\Gamma$ which leads to $\mathcal{E} > \sqrt{\omega V}/l^2\ln(V/\omega)$.

It is interesting to note that, for electrons with the dispersion law $E = p^2/2m$, the energy change is $\Delta E = (p/m)\Delta p$. If $p \sim \Delta p$ then $\Delta p \sim (m\sqrt{D_E t})^{1/2}$. Following the same procedure as before we obtain $D \sim l^{6/5}D_E^{1/5}m^{-2/5} \propto \omega^{2/5}$. However, this model is not very realistic since usually Δp is less than the Fermi momentum, and the estimate (6) should be used.

Finally, we would like to mention another possible way of understanding (2). In this estimate the main point is the connection between the coherence time t_c and the

diffusion rate in energy D_E. Since t_c is the typical time scale of the physical problem, its relation with D_E is quite similar to the relation between the Kolmogorov-Sinai entropy h and the diffusion rate $1/t_c \sim h \propto D_E^{1/3}$ found in plasma physics for the problem of a charged particle in a random magnetic field [16]. Indeed, the equations studied in [16] for the computation of h are close to those obtained for the model (1) written for the ψ-function amplitudes C_m in the instant time eigenstate basis [1]:

$$i\dot{C}_m = \dot{\epsilon} \sum_{r \neq 0} \frac{(\partial \hat{H}/\partial \epsilon)_{m,m-r}}{\tilde{E}_m - \tilde{E}_{m-r}} e^{i(\varphi_m - \varphi_{m-r})} C_{m-r} \ , \qquad (7)$$

where $\hat{H}$ is the Hamiltonian. The expansion coefficients of the states $\psi(t)$ are defined by $\psi(t) = \sum_m C_m \exp[-i\varphi_m(t)]|m(\epsilon)\rangle$ and $\varphi_m(t) = \int dt \tilde{E}_m[\epsilon(t)]$. Due to localization the contribution in the sum (7) is given only by those elements with $|r| < l$. Except for i in the left-hand side, Eqs. (7) are similar to the linearized equations for the calculation of Lyapunov exponents in dynamical systems [16,17]. There it was shown that $h \propto \dot{\epsilon}^{2/3}$, which determines, in agreement with (2), the typical time scale t_c.

In conclusion, we have found a mechanism which allows us to explain the behavior of electrons in a random potential under slow field modulation. This modulation, for small field frequencies, leads to a relatively fast electron diffusion along the lattice. Such diffusion is in turn caused by the Mott mechanism of energy absorption, which leads to the destruction of phase coherence and to delocalization. The diffusion along the lattice also gives an induced dc conductivity at zero temperature, $\sigma_{\rm dc} \propto \omega^{2/3}$. It would be interesting to test this frequency dependence of dc conductivity in laboratory experiments.

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